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# ON THE EXISTENCE OF GENERALIZED COMPLEX SPACE FORMS

BY
ZBIGNIEW OLSZAK
*Institute of Mathematics, Technical University of Wrocław, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland*

ABSTRACT
The existence of generalized complex space forms with nonconstant function $h$ is proved.

## 1. Introduction

The notations and terminology used in this paper are the same as that employed by Tricerri and Vanhecke in [6].

Let $(M, J, g)$ be an almost Hermitian manifold. Denote by $\pi_1$ and $\pi_2$ the generalized curvature tensor fields defined by

$$\pi_1(X, Y)Z = g(X, Z)Y - g(Y, Z)X,$$

$$\pi_2(X, Y)Z = g(JX, Z)JY - g(JY, Z)JX + 2g(JX, Y)JZ,$$

where $X, Y, Z \in \mathscr{X}(M)$. $(M, J, g)$ is said to be a generalized complex space form if its Riemannian curvature tensor $R$ satisfies the condition

$$R = f\pi_1 + h\pi_2, \tag{1.1}$$

where $f$ and $h$ are certain smooth functions on $M$.

Let $(M, J, g)$ be a generalized complex space form for which the function $h$ is not identical zero. Tricerri and Vanhecke [6] have proved the following: If $\dim M \geqq 6$, then $(M, J, g)$ is a complex space form, i.e. a Kählerian manifold with constant holomorphic sectional curvature (consequently, $f = h =$ const).

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The same conclusion holds if $\dim M = 4$ and if, additionally, $f = \text{const}$ or $h = \text{const}$. In the case of $\dim M = 4$ it is also proved that $f + h = \text{const}$ and $(M, J, g)$ is Hermitian on the open subset of $M$ on which $h \neq 0$.

In this context they stated the following problem [6, p. 389]: Do there exist 4-dimensional almost Hermitian manifolds with $R = f\pi_1 + h\pi_2$, where $h$ is a nonconstant smooth function?

In the present paper it is shown that the above question has the positive answer. Exactly, in Theorem 1 we prove that certain conformal deformations of 4-dimensional Bochner flat Kählerian manifolds with nonconstant scalar curvature lead to generalized complex space forms with nonconstant function $h$. Theorem 2 says that any generalized complex space form with nonzero at each point and nonconstant function $h$ can be obtained only in that way.

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## 2. Auxiliary formulas

Before we start with our main results recall basic formulas for conformal deformations of almost Hermitian structures which we need in what follows.

Assume that $(M, J, g)$ is an almost Hermitian manifold with $\dim M = 4$. Let $\sigma$ be a smooth function on $M$ and $\tilde{g}$ the Riemannian metric conformally related to $g$ by $\tilde{g} = e^{-\sigma}g$. Let $\omega = d\sigma$ and $B$ be the contravariant field of $\omega$ given by $g(X, B) = \omega(X)$. It is classical that the Levi-Civita connections generated by $\tilde{g}$ and $g$ are connected by

$$\tilde{\nabla}_X Y = \nabla_X Y - \tfrac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X, Y)B\}. \tag{2.1}$$

Define a symmetric (0,2)-tensor field $P$ on $M$ by

$$P(X, Y) = (\nabla_X \omega)Y + \tfrac{1}{2}\omega(X)\omega(Y) - \tfrac{1}{4}\omega(B)g(X, Y), \tag{2.2}$$

which in view of (2.1) can also be given by

$$P(X, Y) = (\tilde{\nabla}_X \omega)Y - \tfrac{1}{2}\omega(X)\omega(Y) + \tfrac{1}{4}\omega(\tilde{B})\tilde{g}(X, Y), \tag{2.3}$$

where $\tilde{B}$ is the contravariant field of $\omega$ defined by the relation $\tilde{g}(X, \tilde{B}) = \omega(X)$.

Let trace $P$ be the trace of $P$ computed in relation to the metric $g$ (i.e., trace $P = \Sigma\, P(E_i, E_i)$, where $\{E_i, 1 \leqq i \leqq 4\}$ is a frame which is orthonormal with respect to $g$). From (2.2) we find

$$\text{trace}\, P = -\,\delta\omega - \tfrac{1}{2}\omega(B), \tag{2.4}$$

$\delta\omega$ being the codifferential of $\omega$. It is also classical that the Ricci tensors $\tilde{\rho}$ and $\rho$ are connected by

$$\tilde{\rho}(X, Y) = \rho(X, Y) + P(X, Y) + \tfrac{1}{2}(\text{trace } P)g(X, Y), \tag{2.5}$$

and the Ricci $*$-tensors $\tilde{\rho}^*$ and $\rho^*$ by

$$\tilde{\rho}^*(X, Y) = \rho^*(X, Y) + \tfrac{1}{2}\{P(X, Y) + P(JX, JY)\}. \tag{2.6}$$

Consequently, for the scalar curvatures $\tilde{\tau}$ and $\tau$ one has

$$e^{-\sigma}\tilde{\tau} = \tau + 3 \text{ trace } P, \tag{2.7}$$

and for the $*$-scalar curvatures $\tilde{\tau}^*$ and $\tau^*$

$$e^{-\sigma}\tilde{\tau}^* = \tau^* + \text{trace } P. \tag{2.8}$$

## 3. Main results

THEOREM 1. *Let $(M, J, \tilde{g})$ be a Bochner flat Kählerian manifold of dimension 4. Assume, additionally, that the scalar curvature $\tilde{\tau}$ of $\tilde{g}$ is nonzero everywhere on $M$ and nonconstant* (cf. Remark below). *Let $g = e^{\sigma}\tilde{g}$, where $\sigma = -\log(C\tilde{\tau}^2)$, $C = \text{const} > 0$. Then the Hermitian manifold $(M, J, g)$ is a generalized complex space form for which the function $h \neq 0$ everywhere on $M$ and $h \neq$ const.*

PROOF. Let $\tilde{Q}$ be the Ricci operator defined by $\tilde{g}(\tilde{Q}X, Y) = \tilde{\rho}(X, Y)$, $\tilde{\rho}$ being the Ricci tensor of $\tilde{g}$. By identity (4) proved by the present author in [4], we have for $\tilde{Q}$

$$\begin{aligned} &24\,\tilde{g}(\tilde{Q}^2X, Y) - 4\tilde{\tau}\tilde{g}(\tilde{Q}X, Y) - (6\,\widetilde{\text{trace}}\,\tilde{Q}^2 - \tilde{\tau}^2)\tilde{g}(X, Y) \\ &\quad = 4\{-4\tilde{\nabla}^2_{XY}\tilde{\tau} + (\tilde{\Delta}\tilde{\tau})\tilde{g}(X, Y)\}, \end{aligned} \tag{3.1}$$

where $\widetilde{\text{trace}}$ means the trace of a (0,2)-tensor computed in relation to the metric $\tilde{g}$, $\tilde{\nabla}^2$ is the second covariant derivative defined by $\tilde{\nabla}^2_{XY} = \tilde{\nabla}_X\tilde{\nabla}_Y - \tilde{\nabla}_{\tilde{\nabla}_XY}$ and $\tilde{\Delta}\tilde{\tau} = \widetilde{\text{trace}}(\tilde{\nabla}^2\tilde{\tau})$ is the Laplacian of $\tilde{\tau}$. Note that since $\tilde{Q}$ commutes with $J$, it follows from (3.1) that

$$\tilde{\nabla}^2_{JXJY}\tilde{\tau} = \tilde{\nabla}^2_{XY}\tilde{\tau}. \tag{3.2}$$

As $\tilde{Q}$ has two double eigenvalues at each point of $M$ it satisfies the identity (cf. Derdziński [3], Lemma 4(i))

$$\tilde{g}(\tilde{Q}^2X, Y) = \frac{\tilde{\tau}}{2}\tilde{g}(\tilde{Q}X, Y) - \tfrac{1}{8}(\tilde{\tau}^2 - 2\,\widetilde{\operatorname{trace}}\,\tilde{Q}^2)\tilde{g}(X, Y).$$

This and (3.1) lead to

$$\tilde{\rho}(X, Y) = \frac{1}{4}\left(\tilde{\tau} + 2\frac{\tilde{\Delta}\tilde{\tau}}{\tilde{\tau}}\right)\tilde{g}(X, Y) - \frac{2}{\tilde{\tau}}\tilde{\nabla}^2_{XY}\tilde{\tau}. \tag{3.3}$$

It is a straightforward verification that the tensor field $P$ given by (2.3) can be written here in the form

$$P(X, Y) = -\frac{2}{\tilde{\tau}}\tilde{\nabla}^2_{XY}\tilde{\tau} + \tfrac{1}{4}\omega(\tilde{B})\tilde{g}(X, Y). \tag{3.4}$$

Applying (3.3) and (3.4) into (2.5) we find $\rho = \lambda g$, i.e. $(M, J, g)$ is Einsteinian. One can notice that, in view of (3.3), this fact can also be deduced from Proposition 4 of Derdziński [3].

On the other hand, applying (3.2)–(3.4) and the equality $\tilde{\rho}^* = \tilde{\rho}$ in (2.6), we get $\rho^* = \mu g$, i.e. $(M, J, g)$ is $*$-Einsteinian. Furthermore, $(M, J, g)$ is Bochner flat, since the Bochner curvature tensor is an invariant of a conformal transformation.

Now, by Theorem 12.5 of Tricerri and Vanhecke [6], $(M, J, g)$ is a generalized complex space form. Thus we have the relation (1.1), which implies $h = \frac{1}{48}(3\tau^* - \tau)$. Hence, with the help of (2.7), (2.8) and $\tilde{\tau}^* = \tilde{\tau}$ we get $h = (C/24)\tilde{\tau}^3$. Q.e.d.

THEOREM 2. *Let $(M, J, g)$ be a generalized complex space form of dimension* 4 *for which the function $h \neq 0$ at each point of $M$ and $h \neq$* const. *Let $\tilde{g} = e^{-\sigma}g$, where $\sigma = -\frac{1}{3}\log(C_1h^2)$, $C_1 =$* const $> 0$. *Then $(M, J, \tilde{g})$ is a Bochner flat Kählerian manifold and $\sigma = -\log(C\tilde{\tau}^2)$, $C =$* const $> 0$.

PROOF. As we already know, $(M, J, g)$ is Hermitian. In [6] (eq. (12.8)) it is also shown that

$$3h(\nabla_X J)Y = (JY)(f)X - Y(f)JX, \tag{3.5}$$

if $X$, $Y$ are two unit vectors which define orthogonal holomorphic planes $\{X, JX\}$, $\{Y, JY\}$. Since $(M, J, g)$ is Hermitian, it needs to satisfy (cf. Tricerri and Vaisman [5], eq. (2.2))

$$2(\nabla_X J)Y = g(X, Y)JB - g(X, JY)B - \omega(Y)JX + \omega(JY)X \tag{3.6}$$

for any $X$, $Y \in \mathscr{X}(M)$, where $\omega = \delta\Omega \circ J$ is the Lee form and $B$ is, as usual, the contravariant field of $\omega$, $\Omega$ being the fundamental form defined by $\Omega(X, Y) = g(X, JY)$. Comparing (3.5) and (3.6) we see that $3h\omega(X) = 2X(f)$ for any $X \in \mathscr{X}(M)$. Hence and from $f + h = \mathrm{const}$ we have $3h\omega(X) = -2X(h)$. Consequently, $\omega = -\frac{1}{3}d\log(h^2) = d\sigma$. Thus, $(M, J, \tilde{g})$ is Kählerian and $(M, J, g)$ is globally conformal to $(M, J, \tilde{g})$ (in the sense of Vaisman [7]). Clearly, $(M, J, \tilde{g})$ is Bochner flat.

To finish the proof, by (2.4) and (2.7), we get

$$(C_1 h^2)^{1/3}\tilde{\tau} = \tau - 3(\delta\omega + \tfrac{1}{2}\omega(B)). \tag{3.7}$$

On the other hand, for $(M, J, g)$ we have (cf. Vaisman [8], Th. 3.1)

$$\tau - \tau^* = 2\delta\omega + \omega(B). \tag{3.8}$$

Since (1.1) leads immediately to $\tau = 12(f + h)$ and $\tau^* = 4(f + 5h)$, from (3.7) and (3.8) we obtain $C_1\tilde{\tau}^3 = 24^3 h$. Therefore $\sigma = -\log(C\tilde{\tau}^2)$. Q.e.d.

REMARK. Examples of self-dual (consequently, Bochner flat) Kählerian manifolds of dimension 4 and with nonconstant scalar curvature were constructed by Derdziński in [2]. It must be also added that both our theorems are strictly related to results of Derdziński [1]–[3].

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